Lecture 2: Bond Pricing and Yield Measures

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0.1 Learning Objectives

By the end of this lecture you should be able to:

- Explain the time value of money and calculate future values and present values for cash-flows.
- Compute the price of a bond by discounting expected cash-flows at an appropriate discount rate.
- Describe how bond prices move in response to changes in yield and why the price–yield relationship is convex.
- Relate coupon rate and yield to a bond's trading status (par, premium or discount) and recognise how prices evolve as a bond approaches maturity (pull to par).
- Identify factors—market interest rates, credit risk, liquidity, economic expectations, tax policy and inflation—that cause bond prices to change.

- Describe complications in bond pricing arising from embedded options, floating-rate structures and inverse floaters.
- Explain how bonds are quoted clean and dirty, and compute accrued interest.
- Compute yields using the present-value relationship: current yield, yield to maturity, yield to call, yield to put, yield to sinker and yield to worst, as well as the yield for a portfolio and the cash-flow yield for amortising securities.
- Compute the discount margin for a floating-rate security and explain why this margin differs from conventional yield measures.
- Identify the sources of a bond's total return—coupon interest, reinvestment income and capital gain or loss—and discuss reinvestment risk.
- Use total return and horizon analysis to evaluate the realised performance of a bond or bond strategy under explicit assumptions about reinvestment rates and future yields.
- Measure changes in yield using absolute (basis-point) and relative (percentage) conventions.

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1 1 The Time Value of Money

A fundamental principle underlying all fixed income valuation is the **time value of money**. A dollar received today is worth more than a dollar received in the future because the dollar today can be invested to earn interest. When an investor purchases a bond, the investor is exchanging money today for a stream of future cash flows. To determine whether this exchange is fair, we must be able to compare money received at different points in time. The tool that allows us to do this is present value and future value analysis.

In fixed income markets, virtually every valuation problem can be reduced to one question:

> What is the present value of a set of future cash flows?

To answer this question, we must first understand how money grows over time and how future cash flows are discounted back to the present.

1.1 1.1 Future Value

Suppose an investor invests an amount P_0 today at an interest rate r per period. After one period, the investment grows to

$$P_1 = P_0(1 + r).$$

After two periods, the investment grows to

$$P_2 = P_0(1 + r)^2.$$

After n periods, the investment grows to

$$P_n = P_0(1 + r)^n.$$

This is the **future value** of the investment. The key idea behind this formula is **compound interest**: interest earned each period is reinvested and itself earns interest in future periods. In fixed income markets, compounding plays a central role because coupon payments are often reinvested, and the return on a bond depends not only on coupon payments but also on the interest earned on reinvested coupons.

> Worked Example
>
> **Future Value**
> Suppose a pension fund invests $10,000,000 at an annual interest rate of 9.2% for 6 years.
>
> $$P_6 = 10{,}000{,}000(1.092)^6 = 16{,}956{,}500.$$
>
> The investment grows to approximately $16.96 million after six years.

1.2 1.2 Future Value of an Ordinary Annuity

An **ordinary annuity** is a series of equal payments made at the **end** of each period. Coupon payments on most bonds are an example of an ordinary annuity.

Suppose a payment C is made at the end of each period for n periods, and the interest rate is r. The future value of these payments at time n is

$$FV = C\left(\frac{(1 + r)^n - 1}{r}\right).$$

This formula shows that the future value of an annuity depends on both the number of payments and the interest rate at which the payments are reinvested. This is important in fixed income because the total return from a bond depends heavily on the reinvestment of coupon payments.

Worked Example

Future Value of an Annuity
Suppose an investor receives $100 at the end of each year for 5 years and can reinvest each payment at 6%.

$$FV = 100\left(\frac{(1.06)^5 - 1}{0.06}\right) = 100(5.637) = 563.70.$$

The future value of the payments at the end of 5 years is $563.70.

1.3 1.3 Present Value

Present value is the reverse of future value. Instead of asking how much an investment today will grow to in the future, we ask:

> How much is a future cash flow worth today?

If a payment of F will be received in n periods and the interest rate is r, the present value is

$$PV = \frac{F}{(1+r)^n}.$$

This formula is called **discounting**, and the interest rate r is called the **discount rate**. In fixed income markets, the discount rate reflects the required yield on a bond, which depends on interest rates, credit risk, liquidity, and other factors.

Worked Example

Present Value
Suppose an investor will receive $1,000 in 3 years and the discount rate is 5%.

$$PV = \frac{1000}{(1.05)^3} = 863.84.$$

The present value of $1,000 received in three years is $863.84.

1.4 1.4 Present Value of a Series of Future Cash Flows

Most bonds pay multiple cash flows: periodic coupon payments and the principal at maturity. The present value of a series of future cash flows is simply the sum of the present values of each individual cash flow.

If the cash flows are $CF_1, CF_2, \ldots, CF_n$, then

$$PV = \sum_{t=1}^{n} \frac{CF_t}{(1+r)^t}.$$

This is the fundamental bond pricing formula. Every bond pricing problem is an application of this equation.

> Worked Example
>
> **Present Value of Multiple Cash Flows**
> Suppose a bond pays \$60 in one year, \$60 in two years, and \$1,060 in three years. If the discount rate is 5%:
>
> $$PV = \frac{60}{1.05} + \frac{60}{1.05^2} + \frac{1060}{1.05^3} = 1026.71.$$
>
> This is the price of the bond.

1.5 1.5 Present Value of an Ordinary Annuity

If a bond pays a fixed coupon C each period for n periods, the present value of the coupon payments is the present value of an ordinary annuity:

$$PV = C\left(\frac{1-(1+r)^{-n}}{r}\right).$$

The price of a bond can therefore be written as

$$P = C\left(\frac{1-(1+r)^{-n}}{r}\right) + \frac{F}{(1+r)^n}.$$

This formula shows that a bond is simply the present value of an annuity (coupon payments) plus the present value of a single payment (principal).

1.6 1.6 Present Value When Payments Occur More Than Once per Year

In many bond markets, coupon payments are made **semiannually**, meaning twice per year. Some securities pay quarterly or monthly.

When payments occur more than once per year, we must adjust both the interest rate and the number of periods.

If the annual interest rate is r and there are m payments per year, then:

- Interest rate per period $= \frac{r}{m}$
- Number of periods $= n \times m$

The present value formula becomes

$$PV = \sum_{t=1}^{nm} \frac{CF_t}{(1+\frac{r}{m})^t}.$$

Worked Example

Semiannual Bond Pricing
Suppose a bond pays \$80 per year in coupons, paid semiannually, for 5 years. The required yield is 6%.

- Semiannual coupon = \$40
- Semiannual yield = 3%
- Number of periods = 10

$$P = 40\left(\frac{1-(1.03)^{-10}}{0.03}\right) + \frac{1000}{(1.03)^{10}}.$$

This gives the price of the bond.

1.7 Key Insight

The most important idea in fixed income valuation is:

> The price of any fixed income security is the present value of its expected future cash flows discounted at an appropriate discount rate.

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2 2 Pricing a Bond

The price of a bond is equal to the present value of its expected future cash flows. These cash flows consist of periodic coupon payments and the repayment of principal at maturity. Therefore, a bond can be viewed as a financial instrument that converts a series of future payments into a present value.

If a bond has coupon payment C, face value F, maturity n, and required yield r, the price of the bond is

$$P = \sum_{t=1}^{n} \frac{C}{(1+r)^t} + \frac{F}{(1+r)^n}.$$

This formula shows that bond pricing is simply an application of present value. The required yield used to discount the cash flows reflects market interest rates, credit risk, liquidity, and other factors.

2.1 Pricing Zero-Coupon Bonds

A zero-coupon bond makes only one payment at maturity and does not pay periodic coupons. Therefore, its price is simply the present value of the maturity value.

$$P = \frac{F}{(1+r)^n}.$$

Zero-coupon bonds are particularly useful for understanding the relationship between interest rates and bond prices because there is only one cash flow.

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3 3 Price–Yield Relationship

One of the most important relationships in fixed income markets is the inverse relationship between bond prices and yields. When required yields increase, the present value of future cash flows decreases, and therefore the bond price falls. When required yields decrease, the present value of future cash flows increases, and therefore the bond price rises.

This inverse relationship exists because the bond's cash flows are fixed while the discount rate changes.

The relationship between bond price and yield is not linear; it is curved. This curvature is called **convexity**, and it means that bond prices increase at an increasing rate when yields fall and decrease at a decreasing rate when yields rise.

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4 4 Relationship Between Coupon Rate, Required Yield, and Price

The relationship between the coupon rate and the required yield determines whether a bond sells at par, at a premium, or at a discount.

Coupon Rate vs Yield	Bond Price
Coupon rate = Yield	Price = Par
Coupon rate > Yield	Price > Par (Premium)
Coupon rate < Yield	Price < Par (Discount)

If the coupon rate is higher than the market yield, the bond pays more interest than comparable bonds, so investors are willing to pay more than par. If the coupon rate is lower than the market yield, the bond pays less interest than comparable bonds, so investors will only buy it at a discount.

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5 5 Relationship Between Bond Price and Time if Interest Rates Are Unchanged

If interest rates remain unchanged, the price of a bond will move toward its face value as it approaches maturity. This is known as **pull to par**.

- A premium bond will decline in price over time.
- A discount bond will increase in price over time.
- A bond selling at par will remain at par.

This occurs because the present value of the principal repayment becomes less sensitive to discounting as the bond approaches maturity.

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6 6 Reasons for the Change in the Price of a Bond

Bond prices change for several reasons. The most important factor is changes in market interest rates, but other factors also affect bond prices:

- Changes in the level of interest rates
- Changes in credit risk of the issuer
- Changes in liquidity of the bond
- Changes in tax treatment
- Changes in inflation expectations
- Changes in economic conditions

Understanding these factors is essential for fixed income portfolio management because bond prices can change even if interest rates remain constant.

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7 7 Complications in Pricing Bonds

In practice, bond pricing is more complicated than the basic present value formula suggests. Several issues arise in real-world bond pricing.

7.1 Next Coupon Payment Due in Less Than Six Months

Bonds often trade between coupon payment dates. When this happens, the buyer must compensate the seller for the interest earned since the last coupon payment. Therefore, the pricing formula must account for fractional periods.

7.2 Cash Flows May Not Be Known

For some bonds, the future cash flows are uncertain. Examples include:

- Callable bonds
- Putable bonds
- Mortgage-backed securities
- Floating-rate securities

In these cases, cash flows depend on future interest rates or borrower behavior.

7.3 Determining the Appropriate Required Yield

The required yield depends on several factors:

- Risk-free rate
- Credit risk
- Liquidity risk
- Time to maturity
- Tax considerations

Choosing the correct discount rate is often the most difficult part of bond valuation.

7.4 One Discount Rate Applicable to All Cash Flows

The basic bond pricing formula assumes that all cash flows are discounted at the same interest rate. In reality, different cash flows should be discounted using different interest rates based on their maturity. This leads to the concept of the **term structure of interest rates**, which will be discussed later in the course.

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8 8 Pricing Floating-Rate and Inverse-Floating-Rate Securities

Some bonds do not pay fixed coupons. Instead, their coupon payments are linked to a reference rate such as LIBOR or Treasury rates.

8.1 Price of a Floater

A floating-rate bond has a coupon that resets periodically based on a reference rate plus a fixed spread. Because the coupon adjusts to market interest rates, the price of a floating-rate bond typically remains close to its par value.

The price of a floating-rate bond can be expressed as the present value of its expected future cash flows, where the coupon payments depend on future interest rates.

8.2 Price of an Inverse Floater

An inverse floating-rate bond has a coupon that moves in the opposite direction of interest rates. When interest rates rise, the coupon decreases; when interest rates fall, the coupon increases. Because of this structure, inverse floaters are highly sensitive to interest rate changes and can be very volatile.

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9 9 Price Quotes and Accrued Interest

9.1 Price Quotes

Bond prices are typically quoted as a percentage of par value. For example, a price quote of 98.50 means the bond is selling for 98.50% of its face value.

If the face value is $1,000, the price is

$$0.985 \times 1000 = 985.$$

Bond prices are quoted as **clean prices**, which do not include accrued interest.

9.2 Accrued Interest

When a bond is purchased between coupon payments, the buyer must pay the seller the interest that has accrued since the last coupon payment. This is called **accrued interest**.

The actual amount paid for a bond is called the **dirty price**:

$$\text{Dirty Price} = \text{Clean Price} + \text{Accrued Interest}.$$

Accrued interest is calculated as

$$\text{Accrued Interest} = \text{Coupon Payment} \times \frac{\text{Number of Days Since Last Coupon}}{\text{Number of Days in Coupon Period}}.$$

10 11 Computing the Yield or Internal Rate of Return

10.1 11.1 Yield on Any Investment

The **yield** or **internal rate of return (IRR)** on an investment is the interest rate that makes the present value of the investment's cash-flows equal to its price. Mathematically, for an investment with N future cash-flows CF_t and price P, the yield y satisfies

$$P = \sum_{t=1}^{N} \frac{CF_t}{(1+y)^t}.$$

Solving for y generally requires an iterative procedure because y appears in the denominator of each term 965351366966888†L3643-L3669 . A higher assumed yield lowers the present value; a lower assumed yield raises it. The objective is to find the yield for which the present value equals the price.

10.1.1 Special case: a single cash flow

If the investment has just one future cash flow CF_N in N periods, the yield can be solved algebraically:

$$y = \left(\frac{CF_N}{P}\right)^{1/N} - 1.$$

This formula eliminates the need for trial and error because there is only one cash-flow term 965351366966888†L3643-L3669 .

10.2 11.2 Example: Computing Yield by Trial and Error

Suppose a financial instrument with price \$903.10 promises cash-flows of \$100, \$100, \$100 and \$1,000 in years 1–4. To find its yield:

1. Compute the present value of cash-flows at several interest rates.
2. Adjust the rate until the present value matches \$903.10.

Using a 10 % trial rate gives a present value of \$931.69. A 12 % trial rate produces \$875.71. By interpolation, a rate of 11 % equates the present value to the price 965351366966888†L3718-L3767 . The yield on this investment is therefore **11 %**.

10.3 11.3 Periodic Cash Flows and Annualisation

Cash-flows are often paid more than once per year—for example, semiannual coupons. If there are n compounding periods per year, the yield is the rate per period (y) that satisfies

$$P = \sum_{t=1}^{nN} \frac{CF_t}{(1+y)^t}.$$

For semiannual coupons, doubling y gives the **bond-equivalent yield**; for other frequencies, multiply y by the number of periods in the year. To find the **effective annual yield** corresponding to a periodic rate r_p, compute

$$\text{Effective annual yield} = (1 + r_p)^m - 1,$$

where m is the number of periods per year. For example, a quarterly periodic rate of 2 % implies an effective annual yield of $(1.02)^4 - 1 \approx 8.24\,\%$.

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11 12 Conventional Yield Measures

Bonds are commonly quoted using several yield measures that summarise different aspects of return. Each measure uses the basic present-value relationship but makes different assumptions about cash-flows and reinvestment.

11.1 12.1 Current Yield

The **current yield** relates a bond's annual coupon to its market price:

$$\text{current yield} = \frac{\text{annual coupon}}{\text{price}}.$$

For instance, a 15-year bond with a 7 % coupon and price \$769.42 has a current yield of $70/769.42 \approx 9.10\,\%$. A 4.5 % coupon bond priced at 99.531 (per \$100 par) has a current yield of $4.50/99.531 \approx 4.52\,\%$ 965351366966888†L3896-L3922 . The current yield ignores capital gains or losses and the time value of money; it measures coupon income relative to price.

11.2 12.2 Yield to Maturity (YTM)

The **yield to maturity** is the internal rate of return assuming the bond is held to maturity and all coupon payments are reinvested at the same rate. For a bond with price P, coupon payment C, maturity value M and n semiannual periods, the present-value equation is

$$P = \sum_{t=1}^{n} \frac{C}{(1+y)^t} + \frac{M}{(1+y)^n}.$$

Solving for y (a semiannual rate) generally requires trial and error; doubling y gives the bond-equivalent YTM 965351366966888†L3896-L3973 . YTM incorporates coupon income, reinvestment income and any capital gain or loss realised at maturity.

11.3 12.3 Yield to Call, Yield to Put and Yield to Sinker

Callable bonds allow the issuer to redeem the bond at specified call dates for a predetermined price. The **yield to call** is the interest rate that equates the bond's price to the present value of the coupon payments up to the assumed call date plus the call price. Because multiple call dates may exist, investors often compute yields to first call, next call and to any refunding call.

Similarly, **putable bonds** allow the investor to sell the bond back to the issuer on specified dates at a put price. The **yield to put** equates the price to the present value of cash-flows up to the put date plus the put price. Bonds with sinking-fund schedules retire portions of the issue over time. The **yield to sinker** uses the sinking-fund date and price.

The **yield to worst** is the lowest yield among the yield to maturity, all yields to call and all yields to put. It represents the minimum return an investor might realise if the issuer exercises embedded options.

11.4 12.4 Yield for a Portfolio

The yield of a bond portfolio is not simply the average of the individual yields because each bond has a different pattern of cash-flows. To compute a portfolio yield, aggregate the portfolio's cash-flows and find the discount rate that makes the present value of all cash-flows equal the portfolio's price. For example, a portfolio containing bonds of different maturities and coupon rates requires discounting the combined cash-flow schedule to match the total market value.

11.5 12.5 Cash-Flow Yield and Discount Margin

For amortising securities such as mortgage-backed or asset-backed securities, investors receive principal repayments along with interest. The **cash-flow yield** is the rate that equates the price to the projected cash-flows, including scheduled and unscheduled principal repayments. Estimating the cash-flow yield requires assumptions about prepayments and reinvestment.

Floating-rate securities pay interest linked to a reference rate (e.g., LIBOR) plus a quoted margin. The cash-flows depend on future reference rates, so a conventional YTM cannot be calculated directly. Investors instead calculate the **discount margin**, the margin over the reference rate that equates the price to the present value of the projected cash-flows assuming the reference rate remains constant. This measure estimates the average spread an investor can expect over the reference rate.

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12 13 Sources of a Bond's Return and Reinvestment Risk

A bond's total dollar return comes from three sources:

1. **Coupon interest** – the periodic coupon payments.
2. **Capital gain or loss** – the difference between the sale price (or maturity value) and the purchase price.
3. **Reinvestment income** – interest earned from reinvesting the coupon payments (and, for amortising bonds, reinvested principal).

Conventional yields implicitly assume that coupon payments can be reinvested at the computed yield. The risk that future reinvestment rates will be lower than the yield at purchase is **reinvestment risk**. The longer the maturity and the higher the coupon, the more the bond's total return depends on the reinvestment of coupon payments, and therefore the greater the reinvestment risk. Zero-coupon bonds have no reinvestment risk because all return is realised at maturity.

12.1 13.1 Interest-on-Interest Component

For a non-amortising bond paying semiannual coupon C over n periods and reinvestment rate r, the future value of the coupon payments plus reinvestment income after n periods is

$$C\frac{(1+r)^n-1}{r}.$$

The interest-on-interest component is the difference between this amount and the sum of the coupon payments (nC). For long-maturity, high-coupon bonds, the interest-on-interest component may represent a significant portion of total return.

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13 14 Total Return and Horizon Analysis

The yield to maturity and other conventional yields are **promised yields**: they are realised only if the bond is held to the date used in the calculation and coupon payments can be reinvested at that yield. To evaluate a bond's performance over a specific **investment horizon**, investors compute the **total return**, which explicitly incorporates assumptions about reinvestment rates and the bond's sale price at the end of the horizon.

13.1 14.1 Computing Total Return

To compute the total return over a horizon of h semiannual periods:

1. **Reinvestment income** – For each coupon payment received before the horizon, compute its future value at the assumed reinvestment rate until the horizon. Summing these gives the future value of coupon payments plus reinvestment income.
2. **Projected sale price** – Estimate the bond's price at the horizon by discounting its remaining cash-flows at the expected yield to maturity prevailing at that time.
3. **Total future dollars** – Add the future value of coupon payments to the projected sale price. Subtract the initial purchase price to obtain the dollar return.
4. **Total return** – Solve for the interest rate that equates the purchase price to the total future dollars; double the semiannual rate to express as an annualised return.

This procedure accommodates multiple reinvestment rates. For example, coupon payments received in early periods may be reinvested at one rate and later payments at another. The final projected price depends on expectations for the yield environment at the horizon.

13.2 14.2 Example: Total Return

An investor with a three-year horizon considers a 20-year 8 % bond priced at $828.40 with a yield to maturity of 10 %. The investor expects to reinvest coupons at 6 % and to sell the bond after three years at a yield of 7 % for then-17-year bonds. The calculation proceeds as follows:

1. The bond pays $40 every six months. Assuming 3 % semiannual reinvestment, the future value of coupon payments over six periods is $258.74.

2. The projected sale price in three years (with a 7 % yield) is $1,098.51.
3. Total future dollars are $1,357.25; dividing by the purchase price and solving for the semiannual return gives 8.58 % per half year.
4. Doubling yields an **annualised total return of 17.16 %**, demonstrating how reinvestment and expected sale price affect realised performance.

Horizon analysis is also used in bond-swap decisions. By calculating total returns under different reinvestment and yield scenarios, investors can determine whether swapping one bond for another improves projected performance.

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14 15 Measuring Yield Changes

Yield changes can be measured in two ways:

- **Absolute yield change** – measured in basis points. It is the absolute difference between two yields:

$$\text{absolute change (bp)} = |y_{\text{new}} - y_{\text{old}}| \times 100.$$

- **Relative (percentage) yield change** – calculated as the natural logarithm of the ratio of the new yield to the old yield:

$$\text{relative change (\%)} = 100 \times \ln\left(\frac{y_{\text{new}}}{y_{\text{old}}}\right).$$

For example, if yields rise from 4.45 % to 5.11 %, the absolute change is 66 bp and the relative change is $\ln(5.11/4.45) \times 100 \approx 13.83\,\%$. These measures are useful for comparing yield movements across different sectors and maturities.

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15 Key Points

- Bond prices are determined by discounting expected future cash-flows at a required yield; the time value of money is fundamental to this process.
- Bond prices move inversely to yields and the price–yield relationship is convex; premium bonds decline toward par and discount bonds rise toward par as maturity approaches.
- A bond's trading status (par, premium or discount) depends on the relationship between its coupon rate and required yield.
- Bond prices change primarily due to movements in market interest rates, but credit risk, liquidity, economic expectations, tax policy and inflation also play important roles.
- Bond valuation becomes more complex when securities have embedded options, floating or inverse floating coupons, or amortising structures; quoted prices may exclude (clean) or include (dirty) accrued interest.
- The internal rate of return equates the present value of cash-flows to the price and can be computed for any investment; special cases (single cash-flow) have closed-form solutions 965351366966888†L3643-L3669 .
- Conventional yield measures—current yield, yield to maturity, yield to call, yield to put, yield to sinker and yield to worst—summarise different aspects of return and rely on assumptions about holding periods and reinvestment 965351366966888†L3896-L3973 .
- Portfolio yield is obtained by discounting the combined cash-flows of the portfolio; cash-flow yield and discount margin adapt the yield concept to amortising and floating-rate securities.
- A bond's total return comprises coupon interest, reinvestment income and capital gain or loss; reinvestment risk arises when future rates differ from the original yield.
- Total return and horizon analysis provide more realistic performance measures than yield to maturity because they incorporate explicit reinvestment assumptions and projected sale prices.
- Yield changes can be measured in absolute (basis-point) or relative (percentage) terms; both measures are useful for analysing interest-rate movements.

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16 Suggested Readings

Fabozzi – *Bond Markets, Analysis, and Strategies*:
Chapter 2: Pricing of Bonds – covers time value of money, bond pricing, price–yield relationship, convexity, coupon versus yield, price behaviour, and complications in pricing.
Chapter 3: Measuring Yield – explains how to compute yields, describe various yield measures, discuss sources of return and reinvestment risk, and introduce total return and horizon analysis.